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Quantify entanglement by concurrence hierarchy

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Abstract
We define the concurrence hierarchy as $d-1$ independent invariants under local unitary transformations in d-level quantum system. The first one is the original concurrence defined by Wootters (1998 *Phys. Rev. Lett.* **80** 2245) and Hill and Wootters (1997 *Phys. Rev. Lett.* **78** 5022) in a two-level quantum system and generalized to the d-level pure quantum state case. We propose to use this concurrence hierarchy as a measurement of entanglement. This measurement does not increase under local quantum operations and classical communication.

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1. Introduction

Entanglement plays a central role in quantum computation and quantum information [3]. One of the main goals of the theory of entanglement is to develop measures of entanglement. Several measures of entanglement are proposed and studied according to different aims, including entanglement of formation, entanglement of distillation, entanglement cost, etc [4, 5].

Perhaps one of the most widely accepted measures of entanglement is entanglement of formation E_f which provides a very good measurement of entanglement asymptotically. For a pure bipartite quantum state $\rho = |\Phi\rangle\langle\Phi|$ shared by A and B, entanglement of formation is defined by the von Neumann entropy of the reduced density matrix $E_f(\rho) = -\mathrm{Tr}\rho_A \log \rho_A$, where $\rho_A = \mathrm{Tr}_B\rho$. For a mixed state, the entanglement of formation takes the form

$$E_f(\rho) = \inf \sum_j p_j E_f(\Phi_j) \qquad (1)$$

where the infimum is taken over all pure-state decompositions of $\rho = \sum_j p_j |\Phi_j\rangle\langle\Phi_j|$. For a mixed state, this definition is operationally difficult because it requires finding the minimum average entanglement over all possible pure-state decompositions of the given mixed state. In d-dimensions, the explicit expression of entanglement of formation is only found for several special types of mixed state, for example, the isotropic states [6] and Werner states [7]. However, the explicit formulae have been found for the two-level quantum system by

Wootters and co-workers [1, 2]. Here we briefly introduce the results found by Wootters and co-workers. The entanglement of formation of an arbitrary state ρ is related to a quantity called *concurrence* $C(\rho)$ by a function

$$E_f(\rho) = \epsilon(C(\rho)) = h\left(\frac{1+\sqrt{1-C^2(\rho)}}{2}\right) \tag{2}$$

where $h(x) = -x\log x - (1-x)\log(1-x)$ is the binary entropy function. The entanglement of formation is monotonically increasing with respect to the increasing concurrence. The concurrence is defined by an almost magic formula

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \tag{3}$$

where the λ_i are the square roots of the eigenvalues of $\rho\tilde{\rho}$ in descending order. And $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, where σ_y is the Pauli matrix. For a pure state $|\Phi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$, the concurrence takes the form

$$C(\Phi) = |\langle\Phi|\sigma_y \otimes \sigma_y|\Phi^*\rangle| = 2|\alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10}|. \tag{4}$$

Because of the relation between concurrence and entanglement of formation, we can use the concurrence directly as the measure of entanglement.

One important objective in formulating the measures of entanglement is to find whether a bipartite state is separable or not because the entanglement state has some useful applications, for example, teleportation [8], quantum cryptography by using EPR pairs [9]. In a two-level quantum system, the Peres–Horodeckis [10, 11] criterion is a convenient method. However, the concurrence provides another method. If the concurrence is zero, the quantum state is separable, otherwise it is entangled. For a general mixed state in d-dimensions, we have yet to find an operational method to distinguish separability and entanglement.

For a pure state in d-dimensions, the measure of entanglement is largely solved by entanglement of formation. We can use it to distinguish whether a pure state is separable or not and find the amount of entanglement. However, to completely characterize the entanglement, one quantity does not seem enough. A simple example is [12]

$$\begin{aligned} |\psi\rangle &= 1/\sqrt{2}(|00\rangle + |11\rangle) \\ |\phi\rangle &= \sqrt{x}/\sqrt{2}(|00\rangle + |11\rangle) + \sqrt{1-x}|22\rangle. \end{aligned} \tag{5}$$

When $x \approx 0.2271$ is a root of equation $x^x[2(1-x)]^{1-x} = 1$, the entanglement of formation is equal to 1 for both $|\psi\rangle$ and $|\phi\rangle$. However, they cannot be transformed to each other by local operations and classical communication (LOCC).

Because concurrence provides a measure of entanglement in two-level systems, it is worth generalizing the concurrence to higher dimensions. There are several proposals for the case of pure states [12–17]. Uhlmann generalized the concurrence by considering arbitrary conjugations acting on arbitrary Hilbert spaces [13]. Rungta *et al* generalized the spin-flip operator σ_y to a universal inverter S_d defined as $S_d(\rho) = 1 - \rho$, so the pure state concurrence in any dimension takes the form

$$\begin{aligned} C'(\Phi) &= \sqrt{\langle\Phi|S_{d_1} \otimes S_{d_2}(|\Phi\rangle\langle\Phi|)|\Phi\rangle} \\ &= \sqrt{2\left[1 - \mathrm{Tr}\left(\rho_A^2\right)\right]}. \end{aligned} \tag{6}$$

There is a simple relation between these two generalizations pointed out by Wootters [18]. Another generalization proposed by Albeverio and Fei [15] by using an invariant under local unitary transformations turns out to be the same as that of Rungta *et al* up to a whole factor. They defined the concurrence as

$$C(\Phi) = \sqrt{\frac{d}{d-1}\left[1 - \mathrm{Tr}\left(\rho_A^2\right)\right]}. \tag{7}$$

Let us analyse example (5) again by the generalized concurrence. When $x = 1/3$ is a root of equation $(3x - 1)(x - 1) = 0$, the concurrences of $|\psi\rangle$ and $|\phi\rangle$ are equal. But still $|\psi\rangle$ and $|\phi\rangle$ cannot be transformed to each other by LOCC.

As already noted and conjectured by many researchers, one quantity perhaps is not enough to measure all aspects of entanglement [19–26], see [27] for a review, and the geometric properties of entanglement were investigated in [28]. As for the question of separability, the Peres–Horodeckis [10, 11] criterion is enough for a bipartite two-level quantum system. For higher dimensions, if we want to find whether a bipartite state is entangled, besides the partial transposition operation proposed by Peres [10], we need to find other positive but not completely positive maps. Presently, how to find whether a bipartite state in $C^{d_1} \times C^{d_2}$ is entangled is still an open problem.

In this paper, we propose to use not only the invariant (7) to quantify the entanglement of bipartite pure states but also all symmetric functions of eigenvalues of the reduced density operator as measures of entanglement. The first non-trivial quantity is the original invariant under local unitary transformations (7) which provide a separability criterion. Besides the first non-trivial quantity, other quantities also act as measures of entanglement. These quantities are invariant under local unitary transformations and do not increase under LOCC. By using all of these quantities as measures of entanglement, we can show explicitly the reason in some cases why two bipartite pure states cannot be transferred to each other by LOCC since this will cause an increase in different types of entanglement. We also show a counter example that these quantities are still not enough to completely quantify the entanglement. A general formula to obtain these quantities is also presented.

2. Definition of concurrence hierarchy

In this paper, we propose to use the concurrence hierarchy to quantify the entanglement for d-dimensions. We restrict ourselves to the $C^d \otimes C^d$ bipartite pure state. A general bipartite pure state in $C^d \otimes C^d$ can be written as

$$|\Phi\rangle = \sum_{i,j=0}^{d-1} \alpha_{ij}|ij\rangle \tag{8}$$

with normalization $\sum_{ij} \alpha_{ij}\alpha_{ij}^* = 1$. We define a matrix Λ with entries $\Lambda_{ij} = \alpha_{ij}$. The reduced density matrix can be denoted as $\rho_A = \mathrm{Tr}_B\rho = \Lambda\Lambda^\dagger$. Under a local unitary transformation $U \otimes V$, the matrix Λ is changed to $\Lambda \to U^t\Lambda V$, where the superindex t represents transposition. The reduced density operator is thus transformed to

$$\rho_A \to (U^t\Lambda V)(V^\dagger\Lambda^\dagger U^{t\dagger}) = U^t\Lambda\Lambda^\dagger U^{t\dagger}. \tag{9}$$

In two dimensions, it was pointed out by Linden and Popescu [29] that there is one non-trivial invariant under local unitary transformations $I = \mathrm{Tr}(\Lambda\Lambda^\dagger)^2$. In general d-dimensions, it was pointed out by Albeverio and Fei that there are $d - 1$ independent invariants under local unitary transformations $I_k = \mathrm{Tr}(\Lambda\Lambda^\dagger)^{k+1}$. When $k = 0$, it is just the normalization equation $I_0 = \sum_{ij} \alpha_{ij}\alpha_{ij}^* = 1$. For $k = 1, \ldots, d - 1$, I_k are $d - 1$ independent invariants under local unitary transformations. Then they generalize the concurrence as formula (7) and one relation can be calculated as

$$1 - \mathrm{Tr}\rho_A^2 = I_0 - I_1 = \frac{1}{2}\sum_{i,j,k,m}^{d} |\alpha_{ik}\alpha_{jm} - \alpha_{im}\alpha_{jk}|^2. \tag{10}$$

When $C(\Phi) = 0$, it is separable; when $C(\Phi) \neq 0$, it is entangled; when $C(\Phi) = 1$, it is a maximally entangled state. For a pure state $|\Phi\rangle$ as in (8), when $\alpha_{ik}\alpha_{jm} = \alpha_{im}\alpha_{jk}$ for all

i, j, k, m, it can be written as a product form and thus separable. It is a rather intuitive idea to use quantity (10) as the measure of entanglement. Indeed all proposals of generalization of concurrence lead to this result. Also when $C(\Phi) \neq 0$, state $|\Phi\rangle$ is entangled. However, our opinion is that this quantity is necessary but not enough. For quantifying the entanglement, it is dealt with independently by restricting to every two-level system. For example, suppose $|\Phi'\rangle$ takes the form

$$|\Phi'\rangle = \alpha_{00}|00\rangle + \alpha_{11}|11\rangle + \alpha_{22}|22\rangle. \tag{11}$$

Actually we can always change a pure state $|\Phi\rangle$ to this form by Schmidt decomposition. The states $\alpha_{00}|00\rangle + \alpha_{11}|11\rangle$, $\alpha_{00}|00\rangle + \alpha_{22}|22\rangle$ and $\alpha_{11}|11\rangle + \alpha_{22}|22\rangle$ are considered independently in (10) and the entanglement in every two-level system is summed together, $C(\Phi) = |\alpha_{00}\alpha_{11}|^2 + |\alpha_{00}\alpha_{22}|^2 + |\alpha_{11}\alpha_{22}|^2$. As already pointed out, when $x = 1/3$, the concurrences of $|\psi\rangle$ and $|\phi\rangle$ in (5) are equal but they cannot be transformed to each other by LOCC. Our idea here is that besides the concurrence in the form (10), we should also quantify it by other quantities. For example, for the state $|\Phi'\rangle$ in (11), we can quantify the entanglement by

$$C_3(\Phi') = |\alpha_{00}\alpha_{11}\alpha_{22}|^2 \tag{12}$$

up to a normalized factor. In this quantity we just consider the entanglement in all three levels. Apparently, $C_3(\Phi') = 0$ does not mean that the state $|\Phi'\rangle$ is separable. So both this quantity and (10) are necessary in quantifying the entanglement in a three-level quantum system. We call these two quantities the *concurrence hierarchy* for a three-level system. Example (5) thus can be distinguished as follows. If you let both $C(\psi) = C(\phi)$ and $C_3(\psi) = C_3(\phi)$, we can find just one solution $x = 1$, i.e. $|\psi\rangle = |\phi\rangle$. In the case of $x = 1/3$, though the two-level concurrences defined in (7) for $|\psi\rangle$ and $|\phi\rangle$ are equal, their three-level concurrences are different, $C_3(\psi) = 0$ while $C_3(\phi) = 1/54$. The structure of their concurrence hierarchy is different. So, they cannot be transformed to each other by LOCC.

Next, we give our precise definition of concurrence hierarchy. Suppose a bipartite pure state (8) shared by A and B, $\lambda_\Phi = \left\{\lambda_0^\downarrow, \ldots, \lambda_{d-1}^\downarrow\right\}$, denotes the vector of eigenvalues of the reduced density operator $\rho_A = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$ in decreasing order. In other words $\lambda_j^\downarrow$, $j = 0, \ldots, d-1$ are squares of singular values of matrix Λ.

Definition. *The concurrence hierarchy of the state $|\Phi\rangle$ is defined as*

$$\begin{aligned} C_k(\Phi) &= \sum_{0\leqslant i_0<i_1<\cdots<i_k\leqslant(d-1)} \lambda_{i_0}^\downarrow\lambda_{i_1}^\downarrow\cdots\lambda_{i_k}^\downarrow \\ k &= 1, 2, \ldots, d-1. \end{aligned} \tag{13}$$

We propose to use this concurrence hierarchy to quantify the entanglement of the state $|\Phi\rangle$.

The first-level concurrence is trivial since it is just the normalization condition $C_1(\Phi) = \sum_{i=0}^{d-1} \lambda_i^\downarrow = 1$. The two-level concurrence is the d-dimensional generalization of concurrence proposed by Rungta *et al* [14] and Albeverio *et al* [15] up to a whole factor. In two dimensions, there is just one non-trivial concurrence which is the original concurrence proposed by Wootters *et al* [1, 2]. In d-dimensions, the concurrence hierarchy consists of $d-1$ independent non-trivial concurrences. We remark that $C_2(\Phi)$ is enough to characterize the separability. However, the hierarchy of $C_k(\Phi)$ concerns the entanglement transformation. The result of three-level concurrence in three dimensions is already presented in (12). This concurrence hierarchy is invariant under local unitary transformations and can be represented in terms of invariants $I_k = \mathrm{Tr}(\Lambda\Lambda^\dagger)^{k+1}$ [15]. It should be noted that a similar idea to that in this paper was also proposed by Sinolecka *et al* [26]. We give an example to show one relation for three-level concurrence of state $|\Phi\rangle$ in (8),

$$\begin{aligned} C_3(\Phi) &= \sum_{0\leqslant i_0<i_1<i_2\leqslant(d-1)} \lambda^{\downarrow}_{i_0}\lambda^{\downarrow}_{i_1}\lambda^{\downarrow}_{i_2} \\ &= 1 + 2I_2 - 3I_1 \\ &= \frac{1}{6}\sum_{ijklmr} |\alpha_{ij}\alpha_{kl}\alpha_{mr} + \alpha_{kj}\alpha_{ml}\alpha_{ir} + \alpha_{mj}\alpha_{il}\alpha_{kr} - \alpha_{mj}\alpha_{kl}\alpha_{ir} - \alpha_{il}\alpha_{kj}\alpha_{mr} - \alpha_{ij}\alpha_{ml}\alpha_{kr}|^2 \end{aligned} \tag{14}$$

where the terms inside $|\cdot|$ correspond to determinants of the 3×3 submatrix of Λ with row indices i, k, m and column indices j, l, r. When $|\Phi\rangle$ is separable, all concurrences in the hierarchy are zeros except the trivial one. If the Schmidt number (rank) of ρ_A for state $|\Phi\rangle$ in (8) is k, $1 \leqslant k \leqslant d$, all higher level concurrences $C_j(\Phi) = 0$, $j > k$. This is simple because all eigenvalues of ρ_A are non-negative.

3. A simple method to calculate the concurrence hierarchy and entanglement can be quantified by concurrence hierarchy

The concurrence hierarchy can be calculated by its definition (13). The two- and three-level concurrences can be calculated directly by relations (10), (14). Here we show that all concurrences in the hierarchy can be calculated similarly. According to some results in linear algebra (see, for example, [31]), the concurrence hierarchy $C_k(\Phi)$ is equal to the sum of the k-by-k principal minors of reduced density operator $\Lambda\Lambda^\dagger$. However, it is known that these quantities are invariant under unitary transformations $U\Lambda\Lambda^\dagger U^\dagger$. This leads straightforwardly to the result that for a bipartite pure state (8), the concurrence hierarchy $C_k(\Phi)$ is invariant under local unitary transformations. For convenience, we adopt the same notation as in [31]. Let $\beta, \gamma \subseteq \{0, \ldots, d-1\}$ be index sets, each of cardinality k, $k = 1, \ldots, d$. According to Cauchy–Binet formula, we have the following relations:

$$\begin{aligned} C_k(\Phi) &= \sum_{\beta} \det \rho_A(\beta, \beta) \\ &= \sum_{\beta}\sum_{\gamma} \det \Lambda(\beta, \gamma) \det \Lambda^\dagger(\gamma, \beta) \\ &= \sum_{\beta}\sum_{\gamma} |\det \Lambda(\beta, \gamma)|^2 \end{aligned} \tag{15}$$

where we use the relation $\rho_A = \Lambda\Lambda^\dagger$, and the notation $\det \Lambda(\beta, \gamma)$ means the determinant of submatrix Λ with row and column index sets β and γ. When the cardinality $k = 2, 3$, we recover the previous results (10, 14). So, we do not need to calculate the eigenvalues of the reduced density operator to find the concurrence hierarchy, we can calculate the concurrence hierarchy directly by summing the determinants of all k-by-k submatrices of Λ.

Next, we show the concurrence hierarchy cannot increase under LOCC. We use the theorem proposed by Nielsen by majorization scheme [30]. For convenience, we use the same notation as in [31] and Nielsen. The elements of vectors $x = \{x^{\downarrow}_0, \ldots, x^{\downarrow}_{d-1}\}$ and $y = \{y^{\downarrow}_0, \ldots, y^{\downarrow}_{d-1}\}$ are ordered in decreasing order. We say that x is majorized by y, $x \prec y$, if $\sum_{j=0}^{k} x^{\downarrow}_j \leqslant \sum_{j=0}^{k} y^{\downarrow}_j$, $k = 0, \ldots, d-1$ and the equality holds when $k = d-1$.

Theorem 1 [30]. *$|\Psi\rangle$ transforms to $|\Phi\rangle$ using LOCC if and only if λ_Ψ is majorized by λ_Φ,*

$$|\Psi\rangle \to |\Phi\rangle \qquad \textit{iff} \quad \lambda_\Psi \prec \lambda_\Phi. \tag{16}$$

Now we propose our theorem by directly using the Nielsen theorem.

Theorem 2. *$|\Psi\rangle$ transforms to $|\Phi\rangle$ using LOCC, the concurrence hierarchy of $|\Psi\rangle$ is no less than that of $|\Phi\rangle$. And explicitly, if $|\Psi\rangle \to |\Phi\rangle$, then $C_k(\Psi) \geqslant C_k(\Phi), k = 1, \ldots, d$.*

The proof of this theorem is as follows. Because of the Nielsen theorem, $|\Psi\rangle \to |\Phi\rangle$ then we have $\lambda_\Psi \prec \lambda_\Phi$. Because $-C_k, k = 1, \ldots, d$ are isotonic functions [31], i.e. if $\lambda_\Psi \prec \lambda_\Phi$ then $-C_k(\Psi) \leqslant -C_k(\Phi)$. Thus we have $C_k(\Psi) \geqslant C_k(\Phi), k = 1, \ldots, d$. Here we mainly use the fact that each C_k is a Schur-concave function, see [31].

It is well known that a negative entropy function is isotonic, so the entanglement of formation cannot increase under LOCC. Here we show the concurrence hierarchy cannot increase under LOCC.

4. Applications of concurrence hierarchy

According to theorem 2, if some of the relations $C_k(\Psi) \geqslant C_k(\Phi), k = 1, \ldots, d$ do not hold, $|\Psi\rangle$ and $|\Phi\rangle$ cannot be transformed to each other by LOCC. Here we analyse an example raised by Nielsen [30],

$$\begin{aligned} |\Psi\rangle &= \sqrt{0.5}|00\rangle + \sqrt{0.4}|11\rangle + \sqrt{0.1}|22\rangle \\ |\Phi\rangle &= \sqrt{0.6}|00\rangle + \sqrt{0.2}|11\rangle + \sqrt{0.2}|22\rangle. \end{aligned} \tag{17}$$

According to Nielsen theorem, neither $|\Psi\rangle \to |\Phi\rangle$ nor $|\Phi\rangle \to |\Psi\rangle$. Here we analyse this example by calculating their concurrence hierarchy. We can find

$$C_2(\Psi) = 0.29 > C_2(\Phi) = 0.28 \tag{18}$$

$$C_3(\Psi) = 0.020 < C_3(\Phi) = 0.024. \tag{19}$$

It follows from theorem 2 that neither $|\Psi\rangle \to |\Phi\rangle$ nor $|\Phi\rangle \to |\Psi\rangle$. We can roughly interpret the reason as the two-level entanglement of $|\Psi\rangle$ is larger than that of $|\Phi\rangle$ (18), but the three-level entanglement of $|\Psi\rangle$ is less than that of $|\Phi\rangle$ (19). So we cannot transform them to each other by LOCC.

It should be noted that the inverse of theorem 2 is not true. That means even if we have $C_k(\Psi) \geqslant C_k(\Phi), k = 1, \ldots, d$, we are not sure whether $|\Psi\rangle \to |\Phi\rangle$. Here we give an example

$$\begin{aligned} |\Phi'\rangle &= \sqrt{0.5}|00\rangle + \sqrt{0.4}|11\rangle + \sqrt{0.1}|22\rangle \\ |\Psi'\rangle &= \sqrt{0.55}|00\rangle + \sqrt{0.3}|11\rangle + \sqrt{0.15}|22\rangle. \end{aligned} \tag{20}$$

One can find the following relations:

$$C_2(\Psi') = 0.2925 > C_2(\Phi') = 0.29 \tag{21}$$

$$C_3(\Psi') = 0.02475 > C_3(\Phi') = 0.020. \tag{22}$$

According to the Nielsen theorem neither $|\Psi'\rangle \to |\Phi'\rangle$ nor $|\Phi'\rangle \to |\Psi'\rangle$. That means the concurrence hierarchy is not complete. In the sense of classifying pure bipartite states by LOCC, the Nielsen theorem is more powerful. However, our result is mainly to quantify the entanglement by concurrence hierarchy.

5. Summary and discussions

The drawback of the concurrence hierarchy is that it is not complete though the hierarchy consists of $d - 1$ independent invariants. We should note that Vidal [19], Jonathan and Plenio [20], and Hardy [22] found a complete set of entanglement measures consists of $d - 1$ independent entanglement monotones. In concurrence hierarchy, each level of concurrence involves all parameters of a given pure state. So, we can say that each concurrence in the hierarchy describes the entanglement *globally*. For example, $C_2(\Phi)$ describes all two-level entanglements in a pure state $|\Phi\rangle$. If two eigenvalues between $\lambda_\Phi^\downarrow$ and $\lambda_\Phi^\downarrow$ are different, the concurrences in the hierarchy will generally be different.

In summary, we give the definition of concurrence hierarchy and we propose to use the concurrence hierarchy as a measure of entanglement. All concurrences in the hierarchy are zero for separable states except the normalization one. The concurrence hierarchy is invariant under local unitary transformations. The concurrence hierarchy cannot increase by using LOCC. A simple and direct formula (15) is obtained for the concurrence hierarchy. We also analyse some interesting examples by using the concurrence hierarchy.

Our result in this paper is a small step towards completely quantifying the entanglement. However, we find some interesting applications of concurrence hierarchy. It requires much attention to be given along the direction of this paper. We just consider the case of pure states. To study the concurrence hierarchy for mixed states is difficult presently, because even the first non-trivial concurrence of a general mixed state in d-dimensions has not been obtained. We do not even have a widely accepted operational way to find whether a state is entangled. However, our result has potential applications for mixed states. In particular, we give the definition of concurrence hierarchy (13), which could shed light on how we should formulate them for mixed states. We should note that the definition of concurrence hierarchy (13) is just for a pure state. To calculate the concurrence hierarchy for mixed states, we need some formulae like the form of Wootters in two dimensions (3), because we cannot characterize separability only by the eigenvalues of density matrix and reduced density matrices [32].

As we already mentioned, even in the classification of pure states by LOCC, theorem 2 is weaker than the Nielsen theorem though it has interesting applications. But we actually raise an interesting question: both $|\Psi\rangle$ and $|\Phi\rangle$ in (17), and $|\Psi'\rangle$ and $|\Phi'\rangle$ in (20) are incomparable by the Nielsen theorem, however, by using concurrence hierarchy, we show cases (17) and (20) belong to different groups. Then what are the essential differences between cases (17) and (20)?

It is also interesting to consider other series of quantities to quantify entanglement, for example, we can use invariants $I_k = \mathrm{Tr}(\Lambda\Lambda^\dagger)^{k+1}$ as measures of entanglement. Quantum Rényi entropies defined as $S_j = \frac{1}{1-j}\log_2 \mathrm{Tr}(\Lambda\Lambda^\dagger)^j$ (see, for example, [25, 33]) also provide measures of entanglement. Hopefully, quantum Rényi entropies can constitute a complete set of measures of entanglement. And these measures of entanglement work very well for the examples appearing in this paper, i.e., they can determine whether a pure state can be transformed to another by LOCC. However, a proof of whether quantum Rényi entropies is complete or not is necessary. It is also interesting to study whether we can use concurrence hierarchy to study the mixed states, the result in [34] may be useful to this problem. Some results about invariants for multipartite states are already available [35], it is worth studying the corresponding concurrence hierarchy for multipartite states.

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